

# ANSWER TO A QUESTION OF KOLMOGOROV

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**ABSTRACT.** A. N. Kolmogorov asked the following question. Let  $E \subseteq \mathbb{R}^2$  be a measurable set with  $\lambda^2(E) < \infty$ , where  $\lambda^2$  denotes the two-dimensional Lebesgue measure. Does there exist for every  $\varepsilon > 0$  a contraction  $f: E \rightarrow \mathbb{R}^2$  such that  $\lambda^2(f(E)) \geq \lambda^2(E) - \varepsilon$  and  $f(E)$  is a polygon? We answer this question in the negative by constructing a bounded, simply connected open counterexample.

## 1. INTRODUCTION

The following question was posed by M. Laczkovich in [4]. Stand  $\lambda^d$  for the  $d$ -dimensional Lebesgue measure.

**Question 1.1** (M. Laczkovich). *Let  $E \subseteq \mathbb{R}^d$  ( $d \geq 2$ ) be a measurable set with  $\lambda^d(E) > 0$ . Does there exist a Lipschitz onto map  $f: E \rightarrow [0, 1]^d$ ?*

For  $d = 2$  the positive answer to Question 1.1 follows from a result of N. X. Uy [7], and D. Preiss also solved this partial problem by completely different methods. J. Matoušek [5] proved the following stronger, ‘absolute constant’ version based on a well-known combinatorial lemma due to Erdős and Szekeres.

**Theorem 1.2** (J. Matoušek). *There exists a constant  $c > 0$  such that for any measurable set  $E \subseteq \mathbb{R}^2$  with  $\lambda^2(E) = 1$  there exists a 1-Lipschitz onto map  $f: E \rightarrow [0, c]^2$ .*

Question 1.1 is still open for dimensions  $d > 2$ . Theorem 1.2 states that we can contract every set of the plane with positive measure onto a square such that it ‘does not lose too much from its measure’. Can we do this so that the loss of the measure is arbitrarily small? It is impossible with squares as range, but what about polygons? Note that by polygons we mean a wider class of objects than its standard definition does:

**Definition 1.3.** We say that  $P \subseteq \mathbb{R}^2$  is a *polygon* if  $\partial P$  can be covered by finitely many line segments.

The next question is due to A. N. Kolmogorov, it was quoted by P. Alexandroff in a letter written to F. Hausdorff, see [1] and [2].

**Question 1.4** (A. N. Kolmogorov). *Let  $E \subseteq \mathbb{R}^2$  be a measurable set with  $\lambda^2(E) < \infty$ , and let  $\varepsilon > 0$ . Does there exist a contraction  $f: E \rightarrow \mathbb{R}^2$  such that  $\lambda^2(f(E)) \geq \lambda^2(E) - \varepsilon$  and  $f(E)$  is a polygon?*

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The main goal of the paper is to answer Question 1.4 in the negative.

**Theorem 1.5.** *There exist a bounded, simply connected open set  $U \subseteq \mathbb{R}^2$  and  $\varepsilon > 0$  such that if  $f: U \rightarrow \mathbb{R}^2$  is a contraction with  $\lambda^2(f(U)) \geq \lambda^2(U) - \varepsilon$  then  $f(U)$  is not a polygon.*

In contrast to Question 1.1 the higher dimensional versions of Question 1.4 are not more difficult than the original one. The analogue of Theorem 1.5 can be proved similarly for every dimension  $d > 2$  with the straightforward modifications.

The structure of the paper will be as follows. In Section 2 we recall some notation and definitions which we use in this paper. In Section 3 we prove Theorem 1.5. Finally, in Section 4 we collect the open problems.

## 2. PRELIMINARIES

Let  $B(x, r)$  stand for the closed ball of radius  $r$  centered at  $x$ . For a set  $A \subseteq \mathbb{R}^2$  we denote by  $\text{int } A$ ,  $\text{cl } A$  and  $\partial A$  the interior, closure and boundary of  $A$ , respectively. The diameter of a set  $A$  is denoted by  $\text{diam } A$ . The function  $f: A \rightarrow \mathbb{R}^2$  is said to be *Lipschitz* if there exists a constant  $c \in \mathbb{R}$  such that  $|f(x) - f(y)| \leq c|x - y|$  for all  $x, y \in A$ . The smallest such constant  $c$  is called the Lipschitz constant of  $f$  and denoted by  $\text{Lip}(f)$ . If  $\text{Lip}(f) \leq 1$  then  $f$  is a *1-Lipschitz map*, if  $\text{Lip}(f) < 1$  then  $f$  is a *contraction*. If  $A, B \subseteq \mathbb{R}^2$  then let  $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$ .

For the sake of simplicity, stand  $\lambda = \lambda^2$  for the two-dimensional Lebesgue measure. Let us define the 1-dimensional Hausdorff measure as

$$\mathcal{H}^1(A) = \lim_{\delta \rightarrow 0+} \inf \left\{ \sum_{i=1}^{\infty} \text{diam } U_i : A \subseteq \bigcup_{i=1}^{\infty} U_i, \forall i \text{ diam } U_i \leq \delta \right\},$$

for its properties see [6]. Let us denote by  $\underline{0}$  the origin of  $\mathbb{R}^2$ .

## 3. THE PROOF

First we need the following lemma.

**Lemma 3.1.** *Assume  $U \subseteq \mathbb{R}^2$  is a bounded, connected open set and  $f: U \rightarrow \mathbb{R}^2$  is a 1-Lipschitz map such that  $\lambda(f(U)) = \lambda(U)$ . Then  $f$  is an isometry.*

*Proof.* First assume that  $B \subseteq \mathbb{R}^2$  is a closed ball centered at  $x_0$  and  $g: B \rightarrow \mathbb{R}^2$  is a 1-Lipschitz map such that  $\lambda(g(B)) = \lambda(B)$ . We show that  $g$  is an isometry. We may assume by translation that  $g(x_0) = x_0$ . Then  $\text{Lip}(g) \leq 1$  implies  $g(B) \subseteq B$  and  $g(\text{int } B) \subseteq \text{int } B$ . As  $g$  is continuous,  $g(B)$  is compact, so  $B \setminus g(B)$  is relatively open in  $B$ . Therefore  $\lambda(g(B)) = \lambda(B)$  implies  $B \setminus g(B) = \emptyset$ , that is,  $g(B) = B$ . First we prove  $g(\partial B) = \partial B$ . On the one hand,  $g(B) = B$  and  $g(\text{int } B) \subseteq \text{int } B$  yield  $\partial B \subseteq g(\partial B)$ . On the other hand, assume to the contradiction that  $g(\partial B) \not\subseteq \partial B$ . By the continuity of  $g$  there exists a non-degenerate arc  $I \subseteq \partial B$  such that  $\partial B \cap g(I) = \emptyset$ . Then  $\partial B \subseteq g(\partial B)$  yields  $\partial B \subseteq g(\partial B \setminus I)$ . Since the  $\mathcal{H}^1$  measure cannot increase under a 1-Lipschitz map, we obtain

$$\mathcal{H}^1(\partial B) \leq \mathcal{H}^1(g(\partial B \setminus I)) \leq \mathcal{H}^1(\partial B \setminus I) < \mathcal{H}^1(\partial B),$$

that is a contradiction. Hence  $g(\partial B) = \partial B$  follows. Let  $x_1 \in \partial B$  be arbitrary fixed, then  $g(x_1) \in \partial B$ , so we may assume by rotation around  $x_0$  that  $g(x_1) = x_1$ . Let  $x_2$  be the antipodal point of  $x_1$  on  $\partial B$ . Then  $\text{Lip}(g) \leq 1$  and  $g(x_1) = x_1$  imply  $g(B \setminus \{x_2\}) \subseteq B \setminus \{x_2\}$ , so  $g(B) = B$  yields  $g(x_2) = x_2$ . Let us fix  $x_3 \in$

$\partial B \setminus \{x_1, x_2\}$ , and let  $x_4$  be the reflected copy of  $x_3$  to the line determined by  $x_1$  and  $x_2$ . Now  $\text{Lip}(g) \leq 1$  with  $g(x_1) = x_1$  and  $g(x_2) = x_2$  imply  $g(B \setminus \{x_3, x_4\}) \subseteq B \setminus \{x_3, x_4\}$ . Therefore  $g^{-1}(x_3), g^{-1}(x_4) \in \{x_3, x_4\}$ . Therefore we may assume by reflection to the line determined by  $x_1$  and  $x_2$  if necessary that  $g(x_3) = x_3$  and  $g(x_4) = x_4$ . Let  $x \in \partial B$  be arbitrary. Let us choose  $i \in \{1, 2\}$  and  $j \in \{3, 4\}$  such that  $x$  is an element of the shorter arc connecting  $x_i$  and  $x_j$ . Then  $\text{Lip}(g) \leq 1$ ,  $g(x_i) = x_i$  and  $g(x_j) = x_j$  yield  $|g(x) - x_i| \leq |x - x_i|$  and  $|g(x) - x_j| \leq |x - x_j|$ , so  $g(x) \in \partial B$  implies  $g(x) = x$ . Therefore  $g|_{\partial B}$  is the identity. Now let  $x \in B$  be arbitrary. There exists  $a, b \in \partial B$  such that  $x$  is an element of the line segment  $[a, b]$ . Then  $\text{Lip}(g) \leq 1$ ,  $g(a) = a$ , and  $g(b) = b$  yield  $|g(x) - a| \leq |x - a|$  and  $|g(x) - b| \leq |x - b|$ , so  $g(x) = x$ . Therefore  $g$  is the identity.

Finally, let  $U \subseteq \mathbb{R}^2$  be a bounded, connected open set and let  $f: U \rightarrow \mathbb{R}^2$  be a 1-Lipschitz map with  $\lambda(f(U)) = \lambda(U)$ . Let  $x, y \in U$  be arbitrary fixed, it is enough to prove  $|f(x) - f(y)| = |x - y|$ . We prove that for every measurable set  $A \subseteq U$  we have  $\lambda(f(A)) = \lambda(A)$ . Since the Lebesgue measure is additive, subadditive, and cannot increase under a 1-Lipschitz map, we obtain

$$\begin{aligned} \lambda(U) &= \lambda(f(U)) \leq \lambda(f(A)) + \lambda(f(U \setminus A)) \\ &\leq \lambda(A) + \lambda(U \setminus A) = \lambda(U). \end{aligned}$$

Therefore equalities hold in the above inequality, so we have  $\lambda(f(A)) = \lambda(A)$ . The connectedness of  $U$  implies that there are closed balls  $B_1, B_2, \dots, B_k$  such that  $x \in B_1$ ,  $y \in B_k$ , and  $\text{int}(B_i \cap B_{i+1}) \neq \emptyset$  for all  $i \in \{1, \dots, k-1\}$ . As  $\lambda(f(B_i)) = \lambda(B_i)$ , we can apply the first part of the lemma for the maps  $f|_{B_i}$ . This implies that  $f|_{B_i}$  is an isometry for every  $i \in \{1, \dots, k\}$ , therefore  $f|_{\bigcup_{i=1}^k B_i}$  is an isometry, too. Thus  $|f(x) - f(y)| = |x - y|$ , and the proof is complete.  $\square$

Now we are ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* Let  $B$  be the closed unit ball centered at the origin and let  $C \subseteq [0, 1]$  be a nowhere dense compact set with positive one-dimensional Lebesgue measure. Set  $U = \text{int}(B) \setminus (C \times [0, 1])$ . Clearly,  $U$  is open and path-connected. It is easy to see that every simple closed curve can be shrunk to a point continuously in  $U$ , so  $U$  is simply connected. Clearly,  $\text{cl } U = B$  and  $\lambda(U) < \lambda(B)$ .

It is enough to prove that there is an  $\varepsilon > 0$  such that if  $f: U \rightarrow \mathbb{R}^2$  is a contraction with  $\lambda(f(U)) \geq \lambda(U) - \varepsilon$  then  $\lambda(\partial(f(U))) > 0$ . Assume to the contrary that for all  $n \in \mathbb{N}^+$  there are contractions  $g_n: U \rightarrow \mathbb{R}^2$  such that  $\lambda(g_n(U)) \geq \lambda(U) - 1/n$  and  $\lambda(\partial(g_n(U))) = 0$ . Clearly, we may assume that  $\bigcup_{n=1}^{\infty} g_n(U)$  is bounded. Let  $\{z_i : i \in \mathbb{N}\}$  be a dense set in  $U$ . By Cantor's diagonal argument we can choose a strictly increasing subsequence of the positive integers  $\langle n_k \rangle$  such that for every  $i \in \mathbb{N}$  the limit  $\lim_{k \rightarrow \infty} g_{n_k}(z_i)$  exists. Since the maps  $g_{n_k}$  are contractions, the function sequence  $\langle g_{n_k} \rangle$  is uniformly convergent on  $U$ . Therefore we may assume that  $g_n$  converges uniformly to  $g$  for a map  $g: U \rightarrow \mathbb{R}^2$ . The uniform convergence implies that  $g$  is 1-Lipschitz.

First we prove that  $\lambda(g(U)) = \lambda(U)$ . Since  $g$  is a 1-Lipschitz map,  $\lambda(g(U)) \leq \lambda(U)$ , so it is enough to prove the opposite direction. As a continuous image of an open set,  $g(U)$  is  $F_\sigma$ , so measurable. Let  $\delta > 0$  be arbitrary. The regularity of the Lebesgue measure implies that there is an open set  $V$  such that  $g(U) \subseteq V$  and  $\lambda(V) < \lambda(g(U)) + \delta$ . The uniform convergence  $g_n \rightarrow g$  yields that there is an integer  $L$  such that for all  $n > L$  we have  $g_n(U) \subseteq V$ . Therefore the definition of

the maps  $g_n$  imply that for all  $n > L$  we have

$$\lambda(g(U)) + \delta > \lambda(V) \geq \lambda(g_n(U)) \geq \lambda(U) - 1/n.$$

As  $\delta > 0$  is arbitrary, we obtain  $\lambda(g(U)) \geq \lambda(U)$ , so  $\lambda(g(U)) = \lambda(U)$ . Then Lemma 3.1 implies that  $g$  is an isometry. We may assume that  $g$  is the identity, that is,  $g = \text{id}_U$ .

Since  $\text{cl } U = B$ , one can extend the maps  $g_n$  to contractions  $\widehat{g}_n: B \rightarrow \mathbb{R}^2$ . Clearly,  $\widehat{g}_n \rightarrow \text{id}_B$  uniformly on  $B$ . Let  $D \subseteq B$  be a closed ball centered at the origin such that  $\lambda(U) < \lambda(D) < \lambda(B)$ . There exists  $M \in \mathbb{N}$  such that for all  $n > M$  we have

$$(1) \quad \max_{x \in B} |\widehat{g}_n(x) - x| < \text{dist}(D, \partial B).$$

We prove that for all  $n > M$

$$(2) \quad D \subseteq \text{cl}(g_n(U)).$$

Let us fix  $n > M$ . As  $g_n(U)$  is dense in  $\widehat{g}_n(B)$ , we obtain  $\text{cl}(g_n(U)) = \widehat{g}_n(B)$ . Thus we need to prove  $D \subseteq \widehat{g}_n(B)$  for (2). Assume to the contrary that there is an  $x_0 \in D$  such that  $x_0 \notin \widehat{g}_n(B)$ . Set  $r = \text{dist}(D, \partial B)$ . Then (2) implies  $B(x_0, r) \subseteq B$ , so we can define the map  $\phi: B(x_0, r) \rightarrow \mathbb{R}^2$  by  $\phi(x) = -\widehat{g}_n(x) + x + x_0$ . Equation (2) implies  $|\phi(x) - x_0| < r$ , so  $\phi(B(x_0, r)) \subseteq B(x_0, r)$ . Since  $x_0 \notin \widehat{g}_n(B)$ , we obtain that  $\phi(x) \neq x$  for all  $x \in B(x_0, r)$ . Hence  $\phi$  is a continuous self-map of the ball  $B(x_0, r)$  without any fixed points, that contradicts the Brouwer Fixed Point Theorem [3, Proposition 4.4.]. Thus (2) holds.

As the maps  $g_n$  are contraction, we have  $\lambda(g_n(U)) \leq \lambda(U)$ . Therefore  $\lambda(U) < \lambda(D)$  and (2) imply for all  $n > M$ ,

$$\begin{aligned} \lambda(\partial g_n(U)) &\geq \lambda(\text{cl}(g_n(U)) \setminus g_n(U)) \\ &\geq \lambda(\text{cl}(g_n(U))) - \lambda(g_n(U)) \\ &\geq \lambda(D) - \lambda(U) > 0. \end{aligned}$$

Thus  $\lambda(\partial g_n(U)) > 0$ , that contradicts the definition of  $g_n$ . The proof is complete.  $\square$

**Remark 3.2.** It is easy to see that for all Lebesgue null sets  $N \subseteq \mathbb{R}^2$  the sets  $U \Delta N$  are also counterexamples for Question 1.4. On the other hand, one can show that for all  $\varepsilon > 0$  there exist a contraction  $f: U \rightarrow \mathbb{R}^2$  and a Lebesgue null set  $N \subseteq \mathbb{R}^2$  such that  $\lambda^2(f(U)) \geq \lambda^2(U) - \varepsilon$  and  $f(U) \Delta N$  is a polygon. Thus  $U$  will be not a counterexample for Question 4.4.

#### 4. OPEN QUESTIONS

Our most important question is the following.

**Question 4.1.** *Let  $K \subseteq \mathbb{R}^2$  be a compact set, and let  $\varepsilon > 0$ . Does there exist a contraction  $f: K \rightarrow \mathbb{R}^2$  such that  $\lambda^2(f(K)) \geq \lambda^2(K) - \varepsilon$  and  $f(K)$  is a polygon?*

In order to answer Question 4.1 we consider the next question.

**Question 4.2.** *Let  $C \subseteq \mathbb{R}^2$  be a compact set with  $\lambda^2(C) = 0$ , and let  $\varepsilon > 0$ . Does there exist a contraction  $f: C \rightarrow \mathbb{R}^2$  such that  $|f(x) - x| \leq \varepsilon$  for all  $x \in C$  and  $f(C)$  can be covered by finitely many line segments?*

If the compact set  $C$  is a counterexample for Question 4.2 with  $\varepsilon > 0$ , then consider  $K = C \cup R$ , where  $R$  is a closed ring such that the bounded component of its complement contains  $C$ . Then  $K$  is a counterexample for Question 4.1, the sketch of the proof is the following. Assume to the contrary that there are contractions  $f_n: K \rightarrow \mathbb{R}^2$  ( $n \in \mathbb{N}^+$ ) such that  $\lambda(f_n(K)) \geq \lambda(K) - 1/n$  and  $f_n(K)$  is a polygon, that is,  $\partial f_n(K)$  can be covered by finitely many line segments. Similarly as in the proof of Theorem 1.5, one can show that  $f_n$  converges uniformly to an isometry,  $f$ . We may assume that  $f = \text{id}_K$ . Let us fix  $n \in \mathbb{N}^+$  such that  $|f_n(x) - x| \leq \varepsilon$  for all  $x \in C$  and  $f_n(C) \cap f_n(R) = \emptyset$ . As  $f_n$  is a contraction and  $C$  has zero Lebesgue measure,  $f_n(C)$  has also zero measure, so  $\text{dist}(f_n(C), f_n(R)) > 0$  implies  $f_n(C) \subseteq \partial f_n(K)$ . Therefore  $f_n(C)$  can be covered by finitely many line segments, that contradicts the choice of  $C$  and  $\varepsilon$ .

**Remark 4.3.** We do not even know whether the Sierpiński triangle or the Sierpiński carpet is a counterexample for Question 4.2.

Finally, our last question is the following.

**Question 4.4.** Let  $E \subseteq \mathbb{R}^2$  be a measurable set with  $\lambda^2(E) < \infty$ , and let  $\varepsilon > 0$ . Do there exist a contraction  $f: E \rightarrow \mathbb{R}^2$  and a Lebesgue null set  $N \subseteq \mathbb{R}^2$  such that  $\lambda^2(f(E)) \geq \lambda^2(E) - \varepsilon$  and  $f(E) \Delta N$  is a polygon? Is it true at least for compact sets?

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